ON THE CLASSIFICATION OF QUADRATIC HARMONIC MORPHISMS BETWEEN EUCLIDEAN SPACES

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Abstract

We give a classification of quadratic harmonic morphisms between Euclidean spaces (Theorem 2.4) after proving a Rank Lemma. We also find a correspondence between umbilical (Definition 2.7) quadratic harmonic morphisms and Clifford systems. In the case $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$, we determine all quadratic harmonic morphisms and show that, up to a constant factor, they are all bi-equivalent (Definition 3.2) to the well-known Hopf construction map and induce harmonic morphisms bi-equivalent to the Hopf fibration $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$.

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1. Quadratic harmonic morphisms and their equations

Definition 1.1. *A map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is called a* **quadratic map** *if all of the components of φ are quadratic functions (i.e. homogeneous polynomials of degree 2) in $x_1, \dots , x_m$. By a* **quadratic harmonic map** *(respectively a* **quadratic harmonic morphism***) we mean a harmonic map (respectively a harmonic morphism) which is also a quadratic map.*

Note that any quadratic harmonic morphism is a non-constant map by our definition. From the theory of quadratic functions and bilinear forms we know that a quadratic map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ can always be written as

$$\varphi(X) = (X^t A_1 X, \dots , X^t A_n X)$$

where X denotes the column vectors in $\mathbb{R}^m$, X^t the transpose of X and the A_i $(i = 1, \dots , n)$ are symmetric $m \times m$ matrices (henceforth called **component matrices**).

Quadratic harmonic morphisms form a large class of harmonic morphisms between Euclidean spaces as the following examples show.

Example 1.2. *All the following maps are quadratic harmonic morphisms:*

(i) **Quadratic harmonic morphisms from orthogonal multiplications.**
It is well-known that the standard multiplications $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n = 1, 2, 4,$ or 8, in the real algebras of real, complex, quaternionic and Cayley numbers are both orthogonal multiplications and harmonic morphisms. In fact, Baird [1] *(Theorem 7.2.7) proves that these are the only possible dimensions for an orthogonal multiplication $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ to be a harmonic morphism.*

(ii) *(see* [1]*)* **The Hopf construction maps** $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$. *These are defined by*

$$F(X, Y) = (\|X\|^2 - \|Y\|^2 , 2f(X, Y))$$

where f is one of the orthogonal multiplications defined in (i).

(iii) **Quadratic harmonic morphisms from Clifford systems.**
Let $(P_1, ..., P_n)$ be a Clifford system on $\mathbb{R}^{2m}$, i.e. an n-tuple of symmetric endomorphisms of $\mathbb{R}^{2m}$ satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I$ for $i, j = 1, ...n$.

Then it follows from Baird [1]*(Theorem 8.4.1) that*

$$F(X) = (\langle P_1X, X\rangle, \dots, \langle P_nX, X\rangle)$$

(where $\langle , \rangle$ denotes the inner product in Euclidean space) is a quadratic harmonic morphism with dilation $\lambda^2(X) = 4\|X\|^2$ for each $X \in \mathbb{R}^{2m}$.

(*iv*) **Quadratic harmonic morphisms from the complete lifts**. *Let $\varphi : \mathbb{R}^m \supset U \longrightarrow \mathbb{R}^n$ be a C^1 map from an open connected subset of $\mathbb{R}^m$ into $\mathbb{R}^n$. The (real) complete lift (cf.*[8]*Definition 2.1) of φ is the map $\overline{\varphi} : \mathbb{R}^{2m} \supset U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, given by $\overline{\varphi}(X, Y) = J(\varphi(X))Y$, where $J(\varphi(X))$ is the Jacobian matrix of φ at $X \in U$. It follows from Ou* [8] *(Theorem3.3) that* **the complete lift of any quadratic harmonic morphism is again a quadratic harmonic morphism**.

For some further examples, see Loubeau [7]. In the rest of this section we will give equations that characterize quadratic harmonic morphisms between Euclidean spaces.

Lemma 1.3. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic map with $\varphi(X) = (X^tA_1X, \dots, X^tA_nX)$. Then φ is harmonic if and only if*

$$trA_i = 0,\ (i = 1, ..., n). \tag{1}$$

Proof. The harmonicity of φ is equivalent to the statement that all components of φ are harmonic functions, which is easily seen to be equivalent to Equation (1). □

Proposition 1.4. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic map with $\varphi(X) = (X^tA_1X, \dots, X^tA_nX)$. Then φ is horizontally weakly conformal if and only if the following equations hold*

$$A_iA_j + A_jA_i = 0,\ (i, j = 1, ...n,\ i \neq j), \tag{2}$$

$${A_i}^2 = {A_j}^2,\ (i, j = 1, ...n). \tag{3}$$

Proof. For a map $\varphi(X) = (\varphi^1(X), \dots, \varphi^n(X))$ between Euclidean spaces, horizontal weakly conformality is equivalent to (See [4],[6])

$$\langle \nabla\varphi^i(X), \nabla\varphi^j(X)\rangle = \lambda^2(X)\delta^{ij} \tag{4}$$

where δ^{ij} is the Kronecker delta and $\nabla\varphi^i(X)$ denotes the gradient of the component function of $\varphi^i(X)$.

Now for quadratic map φ, we can calculate its Jacobian matrix as

$$J(\varphi(X)) = \begin{pmatrix} 2X^tA_1 \\ \vdots \\ 2X^tA_n \end{pmatrix}.$$

It is easily seen that Equation (4) is equivalent to the following two equations

$$X^tA_iA_jX \equiv 0,\ (i,j = 1, ...n, i \neq j) \tag{5}$$

$$X^t{A_i}^2X \equiv X^t{A_j}^2X,\ (i,j = 1, ...n). \tag{6}$$

Since Equations (5) and (6) are identities of quadratic functions in $x_1, \dots, x_m$, and noting that A_iA_j is not symmetric in general we conclude that (5) and (6) are equivalent to (2) and (3) respectively. Thus we end the proof of the proposition. □

It is well-known (see [4],[6]) that a map between Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and a horizontal weakly conformal map. So by combining Lemma 1.3 and Proposition 1.4 we have

Theorem 1.5. *A quadratic map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m \geq n)$ with $\varphi(X) = (X^tA_1X, \dots, X^tA_nX)$ is a harmonic morphism if and only if*

$$(1)\ trA_i = 0,\ (i = 1, \dots, n),$$
$$(2)\ A_iA_j + A_jA_i = 0,\ (i,j = 1, ...n,\ i \neq j),$$
$$(3)\ {A_i}^2 = {A_j}^2,\ (i,j = 1, ...n).$$

2. THE CLASSIFICATION

In this section we shall prove the Rank Lemma for quadratic harmonic morphisms which will be the basis for the classification theorems.

Lemma 2.1. *(***The Rank Lemma for quadratic harmonic morphisms***)* *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic harmonic morphism with $\varphi(X) = (X^tA_1X, \dots, X^tA_nX)$, Then*
(a) All the component matrices A_i have the same rank which is an even number.
(b) All the component matrices A_i have the same spectrum.

Proof. Suppose that $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a quadratic harmonic morphism with $\varphi(X) = (X^tA_1X, \dots, X^tA_nX)$. Then by Theorem 1.5 we have

$${A_i}^2 = {A_j}^2,\ (i,j = 1, ...n)$$

which implies that

$$\text{rank}{A_i}^2 = \text{rank}{A_j}^2.$$

The equality of $\text{rank}A_i$ now follows from the following
Claim. For any symmetric matrix A, $\text{rank}A^2 = \text{rank}A$.
Proof of Claim. It is a standard fact that A can be diagonalized by an orthogonal matrix P, so $P^{-1}AP = D$ is a diagonal matrix. But

$$\text{rank}A^2 = \text{rank}P^{-1}A^2P = \text{rank}P^{-1}APP^{-1}AP$$
$$= \text{rank}D^2 = \text{rank}D = \text{rank}P^{-1}AP = \text{rank}A.$$

Now we show that $\text{rank}A_i$ is even. It suffices to do the proof for quadratic harmonic morphism $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^2$ with $\varphi(X) = (X^tA_1X, X^tA_2X)$. After a suitable choice of orthogonal coordinates, A_1 assumes the diagonal form

$$A_1 = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & -D_2 & 0 \\ 0 & 0 & 0_r \end{pmatrix} \tag{7}$$

where 0_r denotes the $r \times r$ zero matrix, D_1 is the $k \times k$ diagonal matrix with entries the positive eigenvalues $S_+ = \{\lambda_1, \dots, \lambda_k\}$, and D_2 is the $l \times l$ diagonal matrix with the entries the absolute values of the negative eigenvalues $S_- = \{\xi_1, \dots, \xi_l\})$, where $k + l + r = m$.

Using Equations (2) and (3) we see that A_2 must have the form

$$A_2 = \begin{pmatrix} 0 & B_1 & 0 \\ B_1^t & 0 & 0 \\ 0 & 0 & 0_r \end{pmatrix} \tag{8}$$

where B_1 denotes a $k \times l$ matrix satisfying $D_1B_1 = B_1D_2$, which means

$$\begin{pmatrix} \lambda_1 b_{11} & \dots & \lambda_1 b_{1l} \\ \vdots & \dots & \vdots \\ \lambda_k b_{k1} & \dots & \lambda_k b_{kl} \end{pmatrix} = \begin{pmatrix} \xi_1 b_{11} & \dots & \xi_l b_{1l} \\ \vdots & \dots & \vdots \\ \xi_1 b_{k1} & \dots & \xi_l b_{kl} \end{pmatrix}. \tag{9}$$

Since $\text{rank}A_1 = \text{rank}A_2 = k + l$, we see from Equation (9) that any $\lambda_i \in S_+$ must be equal to one of the numbers in S_- otherwise the ith row of B_1 would be zero vector and $\text{rank}A_2 < k + l$. This means that $S_+ \subset S_-$. A similar reasoning gives $S_- \subset S_+$. Thus we have $S_+ = S_-$, which means that S_+ and S_- have equal numbers of the same elements, i.e. $k = l$ and $S_+ = S_- = \{\lambda_1, \dots, \lambda_k\}$. Thus $\text{rank}A_1 = \text{rank}A_2 = 2k$ is even, which ends the proof of (a). For (b) we first note, from the above proof, that the eigenvalues of a component matrix of a quadratic harmonic morphism must

appear in pairs $\pm\lambda$. On the other hand, it is elementary that if a symmetric linear transformation A^2 has an eigenvalue $\lambda^2 > 0$ then A must have one of eigenvalues $\pm\lambda$. Now the rest of the proof follows from the fact that all $A_i^!2$ $(i = 2, \ldots, n)$ have eigenvalues $\{\lambda_1^2, \ldots, \lambda_k^2\}$, where $\lambda_1, \ldots, \lambda_k$ are the positive eigenvalues of A_1. $\square$

Definition 2.2. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic harmonic morphism. Then the Q-**rank of** φ, denoted by Q-rank(φ), is defined to be the rank of its component matrices. φ is said to be Q-**nonsingular** if Q-rank$(\varphi) = m$, otherwise it is said to be Q-**singular**.*

We are now ready to give a characterization of quadratic harmonic morphisms to $\mathbb{R}^2$.

Proposition 2.3. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^2$ $(m \geq 2)$ be a quadratic harmonic morphism.*
(i) If φ is Q-nonsingular, then $m = 2k$ for some $k \in \mathbb{N}$ and, with respect to suitable orthogonal coordinates in $\mathbb{R}^m$, φ assumes the normal form

$$\varphi(X) = \left(X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X,\ X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X \right)$$

where D, $B_1 \in GL(\mathbb{R}, k)$, with D diagonal and satisfying

$$\begin{cases} DB_1 = B_1 D \\ B_1^t B_1 = D^2. \end{cases}$$

(ii) Otherwise Q-rank$(\varphi) = 2k$, for some k, $0 \leq k < m/2$, and φ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a Q-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^2$.

Proof. Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^2$ be given by $\varphi(X) = (X^t A_1 X, X^t A_2 X)$. Then from the Rank Lemma we know that Q-rank(φ) is even. If φ is Q-nonsingular then Q-rank$(\varphi) = m = 2k$. As in the proof of the Rank Lemma, after a suitable choice of orthogonal coordinates, A_1 assumes the normal form

$$A_1 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \tag{10}$$

where D denotes the $k \times k$ diagonal matrix having the positive eigenvalues of A_1 as diagonal entries. Then A_2 must have the form

$$A_2 = \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix}$$

with $B_1 \in GL(\mathbb{R}, k)$ satisfying $DB_1 = B_1 D$. This, together with $B_1^t B_1 = D^2$ given by (3) of Theorem 1.5, gives (i). Now (ii) follows from the fact that if φ is Q-singular with rank$A = 2k < m$, then after a suitable choice of orthogonal coordinates, A takes the form (7) and consequently B the form (8).

□

Now we give the Classification Theorem for general quadratic harmonic morphisms.

Theorem 2.4. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m \geq n)$ be a quadratic harmonic morphism.*
(I) If φ is Q-nonsingular, then $m = 2k$ for some $k \in \mathbb{N}$ and, with respect to suitable orthogonal coordinates in $\mathbb{R}^m$, φ assumes the normal form

$$\varphi(X) = \left(X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X,\ X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X, \dots ,\right.$$
$$\left. X^t \begin{pmatrix} 0 & B_{n-1} \\ B_{n-1}^t & 0 \end{pmatrix} X \right). \tag{11}$$

where $D, B_i \in GL(\mathbb{R}, k)$ with D diagonal having the positive eigenvalues as its diagonal entries satisfy

$$\begin{cases} DB_i = B_i D \\ B_i^t B_i = D^2 \\ B_i^t B_j = -B_j^t B_i. \ (i, j, = i, ..., n-1, i \neq j). \end{cases} \tag{12}$$

(II) Otherwise Q-rank$(\varphi) = 2k$ for some k, $0 \leq k < m/2$, and φ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a Q-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$.

Proof. As in the proof of the Rank Lemma, after a suitable choice of orthogonal coordinates the first component matrix has the form (7), and all the other component matrices A_n has the form

$$A_{i+1} = \begin{pmatrix} 0 & B_i & 0 \\ B_i^t & 0 & 0 \\ 0 & 0 & 0_r \end{pmatrix},\ i = 1, \dots , n-1.$$

Now if φ is Q-singular then Q-rank$(\varphi) = 2k < m$, for some k, $0 \leq k < m/2$. It is easily seen that φ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a Q-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$. Otherwise φ is Q-nonsingular in which case $r = 0$. Thus we have the normal form (11). Note that for $n > 2$ Equation (2) gives the additional Equation (12). □

Corollary 2.5. *Any quadratic harmonic morphism is the composition of an orthogonal projection followed by a Q-nonsingular quadratic harmonic morphism from an even-dimensional space.*

Remark 2.6. *Thus to study quadratic harmonic morphisms it suffices to consider Q-nonsingular ones from even-dimensional spaces.*

Definition 2.7. *A quadratic harmonic morphism $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $\varphi(X) = (X^t A_1 X, \dots , X^t A_n X)$ is said to be* **umbilical** *if all the positive eigenvalues of one (and hence all by The Rank Lemma) of its component matrices are equal.*

There do exist quadratic harmonic morphisms which are not umbilical as the following example shows.

Example 2.8. *It can be checked that $\varphi : \mathbb{R}^8 \longrightarrow \mathbb{R}^3$ given by*

$$\begin{aligned}\varphi =&(2x_1^2 + 2x_2^2 + 3x_3^2 + 3x_4^2 - 2x_5^2 - 2x_6^2 - 3x_7^2 - 3x_8^2,\\ &4x_1x_5 + 4x_2x_6 + 6x_3x_8 - 6x_4x_7,\\ &-4x_1x_6 + 4x_2x_5 + 6x_3x_7 + 6x_4x_8)\end{aligned}$$

is a quadratic harmonic morphism which is not umbilical since its component matrices have two distinct positive eigenvalues.

For more results on constructions of harmonic morphisms into Euclidean spaces see Ou [9].

3. Quadratic harmonic morphisms and Clifford systems

Definition 3.1. *i) The $(n + 1)$-tuple $(P_0, \dots , P_n)$ of symmetric endomorphisms of $\mathbb{R}^{2m}$ is called a* **Clifford system** *on $\mathbb{R}^{2m}$ if*

$$P_iP_j + P_jP_i = 2\delta_{ij}I \ \ (i, j = 0, 1, ..., n).$$

ii) Let $(P_0, \dots , P_n)$ and $(Q_0, \dots , Q_n)$ be Clifford systems on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q}$ respectively, then $(P_0 \oplus Q_0, \dots , P_n \oplus Q_n)$ is a Clifford system on $\mathbb{R}^{2p+2q}$, the so-called **direct sum** *of $(P_0, \dots , P_n)$ and $(Q_0, \dots , Q_n)$.*
iii) A Clifford system $(P_0, \dots , P_n)$ on $\mathbb{R}^{2m}$ is called **irreducible** *if it is not possible to write $\mathbb{R}^{2m}$ as a direct sum of two non-trivial subspaces which are invariant under all P_i.*
iv) Two Clifford systems $(P_0, \dots , P_n)$ and $(Q_0, \dots , Q_n)$ on $\mathbb{R}^{2m}$ are said to be **algebraically equivalent** *if there exists $A \in O(\mathbb{R}^{2m})$ such that $Q_i = AP_iA^t$ for all $i = 0, 1, \dots , n$.*

From the representation theory of Clifford algebras (see [5]) we have the following results:
Theorem A.(See [3])
(a) Each Clifford system is algebraically equivalent to a direct sum of irreducible Clifford systems.
(b) An irreducible Clifford system $(P_0, \ldots, P_n)$ on $\mathbb{R}^{2m}$ exists precisely for the following values of n and $m = \delta(n)$:

n	1	2	3	4	5	6	7	8	...	n+8
$\delta(n)$	1	2	4	4	8	8	8	8	...	16 $\delta(n)$

(c) For $n \not\equiv 0 \; mod\, 4$, there exists exactly one algebraically equivalent class of irreducible Clifford systems. If $n \equiv 0 \; mod\, 4$, there are two.

Definition 3.2. *Let $\varphi, \tilde{\varphi} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be two quadratic harmonic morphisms. Then*
(1) φ and $\tilde{\varphi}$ are said to be **domain-equivalent** *if there exists an isometry P of $\mathbb{R}^m$ such that $\varphi = \tilde{\varphi} \circ P$. They are said to be* **bi-equivalent** *if there exist isometries P of $\mathbb{R}^m$ and G of $\mathbb{R}^n$ such that $\varphi = G^{-1} \circ \tilde{\varphi} \circ P$.*
(2) The concepts of domain-equivalence *and* bi-equivalence *can be defined similarly for harmonic morphisms between spheres (or, indeed any Riemannian manifolds).*

Baird has proved ([1] Theorem 8.4.1) that any Clifford system $(P_0, \ldots, P_n)$ on $\mathbb{R}^{2m}$ gives rise to a quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{n+1}$ defined by

$$\varphi(X) = (\langle P_0X, X\rangle,\ \langle P_1X, X\rangle, \ldots, \langle P_nX, X\rangle).$$

It is easy to see that two Clifford systems $(P_0, \ldots, P_n)$ and $(Q_0, \ldots, Q_n)$ on $\mathbb{R}^{2m}$ are algebraically equivalent if and only if they give rise to domain-equivalent quadratic harmonic morphisms $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{n+1}$. It is easy to see that any quadratic harmonic morphism given by a Clifford system is umbilical. We shall prove that up to a constant factor all umbilical quadratic harmonic morphisms arise this way.

Theorem 3.3. *Up to a homothetic change of coordinates in $\mathbb{R}^m$, any umbilical quadratic harmonic morphism $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ arises from a Clifford system.*

Proof. We need only to show that, up to a homothetic change of the coordinates in $\mathbb{R}^{2k}$, the component matrices of any Q-nonsingular umbilical

quadratic harmonic morphism $\varphi : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$ represent a Clifford system. Indeed it follows from Theorem 2.4 that with respect to suitable orthogonal coordinates in $\mathbb{R}^{2k}$, φ assumes the normal form (11) with $D = \lambda Id$, and it is easily seen that after a change of scale in $\mathbb{R}^{2k}$ the component matrices become

$$A_1 = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & \tilde{B}_1 \\ \tilde{B}_1^t & 0 \end{pmatrix},\ \dots, A_n = \begin{pmatrix} 0 & \tilde{B}_{n-1} \\ \tilde{B}_{n-1}^t & 0 \end{pmatrix}$$

with $\tilde{B}_i \in O(k)$ satisfying $\tilde{B}_i^t \tilde{B}_j = -\tilde{B}_j^t \tilde{B}_i$. $(i, j, = i, ..., n-1, i \neq j)$. It can be checked that

$$A_\alpha A_\beta + A_\beta A_\alpha = 2\delta_{\alpha\beta} I,\ \ (\alpha, \beta = 1, ..., n).$$

Which means that the A_α represent a Clifford system. This ends the proof of the theorem. $\square$

Example 3.4. *It is easy to check that* $\varphi : \mathbb{R}^8 \longrightarrow \mathbb{R}^5$ *given by*

$$\begin{aligned}\varphi(x, y) = (3|x|^2 - 3|y|^2,\ & 6x_1y_1 - 6x_2y_2 - 6x_3y_3 - 6x_4y_4,\\ & 6x_1y_2 + 6x_2y_1 + 6x_3y_4 - 6x_4y_3,\\ & 6x_1y_3 + 6x_3y_1 + 6x_4y_2 - 6x_2y_4,\\ & 6x_1y_4 + 6x_4y_1 - 6x_2y_3 - 6x_3y_2).\end{aligned}$$

is an umbilical quadratic harmonic morphism with all positive eigenvalues equal to 3*. It is also easy to see that it arises from a Clifford system.*

In the rest of this section we will determine all quadratic harmonic morphisms from $\mathbb{R}^4$ to $\mathbb{R}^3$ and show that they are all bi-equivalent to some constant multiple $\lambda\varphi_0$ of the standard Hopf construction map and that, up to a change of scale, they all restrict to $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and hence induce bi-equivalent harmonic morphisms. We thus recover, by simple means, part of a result of Eells and Yiu [2]. For further results on the existence of quadratic harmonic morphisms see Ou [10]

Theorem 3.5. *Up to* **domain-equivalence***, all quadratic harmonic morphisms* $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ *are of the form*

$$\begin{aligned}\varphi_t =& \lambda(x_1^2 + x_2^2 - x_3^2 - x_4^2,\\ & 2x_1x_3\cos t + 2x_1x_4\sin t - 2x_2x_3\sin t + 2x_2x_4\cos t,\\ & 2x_1x_3\sin t - 2x_1x_4\cos t + 2x_2x_3\cos t + 2x_2x_4\sin t)\end{aligned} \tag{13}$$

where $\lambda \neq 0$ *and* $t \in [0, 2\pi)$*. They are all* **bi-equivalent** *to a constant multiple of the standard Hopf construction map:*

$$\lambda\varphi_0 = \lambda(x_1^2 + x_2^2 - x_3^2 - x_4^2,\ 2x_1x_3 + 2x_2x_4, -2x_1x_4 + 2x_2x_3). \tag{14}$$

Proof. First we note that any quadratic harmonic morphism $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is Q-nonsingular since otherwise, φ would be of the form

$$\mathbb{R}^4 \xrightarrow{\pi} \mathbb{R}^2 \xrightarrow{\varphi_1} \mathbb{R}^3$$

where φ_1 is a non-constant quadratic harmonic morphism which is impossible. Next we
Claim: All quadratic harmonic morphisms $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ are umbilical.
Proof of Claim: Let $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a quadratic harmonic morphism. Then from Theorem 2.4 we have

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in GL(\mathbb{R}, 2)$$

satisfying Equation (12). Now suppose that $\lambda_1 \neq \lambda_2$, then by using the first equation of (12) we have

$$B_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \in GL(\mathbb{R}, 2).$$

But then the third equation of (12) gives $a_1 b_1 = 0$ and $a_2 b_2 = 0$, which is impossible since B_1, B_2 are invertible. Thus we must have $\lambda_1 = \lambda_2$.
Now any Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ can be assumed to be of the form
(15)

$$\varphi = \lambda \left(x_1^2 + x_2^2 - x_3^2 - x_4^2, \;\; X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X, \;\; X^t \begin{pmatrix} 0 & B_2 \\ B_2^t & 0 \end{pmatrix} X \right)$$

where $B_1,\ B_2 \in O(2)$ satisfy

$$B_1^t B_2 = -B_2^t B_1. \tag{16}$$

without loss of generality, we may assume that $B_1 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2)$, and $B_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$, or $B_2 = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \in O(2)\backslash SO(2)$. It can be checked that for the second possibility, Equation (16) has no solution. whilst for the first possibility Equation (16) is equivalent to

$$\begin{cases} \cos(t-\theta) = -\cos(\theta - t) \\ \sin(t-\theta) = -\sin(\theta - t) \end{cases} \tag{17}$$

which has solutions $t - \theta = \theta - t \;\; mod\ 2\pi$ i.e.,

$$\theta = t - \frac{\pi}{2}\ (mod\ \pi) = t \pm \frac{\pi}{2}\ (mod\ 2\pi) \tag{18}$$

Inserting (18) into (15) we have two families

$$
\begin{aligned}
(a)\quad \varphi_t = \lambda(&x_1^2 + x_2^2 - x_3^2 - x_4^2,\\
&2x_1x_3\cos t + 2x_1x_4\sin t - 2x_2x_3\sin t + 2x_2x_4\cos t,\\
&2x_1x_3\sin t - 2x_1x_4\cos t + 2x_2x_3\cos t + 2x_2x_4\sin t)
\end{aligned}
$$

and

$$
\begin{aligned}
(b)\quad \varphi_t = \lambda(&x_1^2 + x_2^2 - x_3^2 - x_4^2,\\
&2x_1x_3\cos t + 2x_1x_4\sin t - 2x_2x_3\sin t + 2x_2x_4\cos t,\\
&-2x_1x_3\sin t + 2x_1x_4\cos t - 2x_2x_3\cos t - 2x_2x_4\sin t).
\end{aligned}
$$

However, family (b) can be obtained from family (a) by an orthogonal change of coordinates in $\mathbb{R}^3$. Thus any Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is domain-equivalent to some φ_t, whilst $\varphi_t = G^{-1} \circ \lambda\varphi_0$ for G given by

$$
G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \in SO(3). \tag{19}
$$

therefore any Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is bi-equivalent to a multiple of the Hopf construction map $\lambda\varphi_0$. This ends the proof of the theorem. □

Corollary 3.6. *Up to homothety of $\mathbb{R}^4$, all quadratic harmonic morphisms $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ arise from algebraically equivalent irreducible Clifford systems on $\mathbb{R}^4$, and they induce harmonic morphisms $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$ bi-equivalent to the standard Hopf fibration given by (14) with $\lambda = 1$.*

Proof. It is trivial to check that for all t, the Clifford systems on $\mathbb{R}^4$ represented by

$$
A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \\ \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \end{pmatrix},
$$

$$
A_2 = \begin{pmatrix} 0 & 0 & \sin t & -\cos t \\ 0 & 0 & \cos t & \sin t \\ \sin t & \cos t & 0 & 0 \\ -\cos t & \sin t & 0 & 0 \end{pmatrix},
$$

are irreducible and are clearly algebraically equivalent. We have seen that, after a possible change of scale in $\mathbb{R}^4$, $\varphi_t = G^{-1} \circ \varphi_0$ for G given by (19).

Thus for all t, φ_t restricts to $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and is bi-equivalent to the classical Hopf fibration $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$, which ends the proof of the Corollary. $\square$

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